BICUBIC INTERPOLATION

HOXIDE MA

ABSTRACT. I worte it to describe my BiCubic Interpolation program in Octave.

Bicubic interpolation is a common method of interpolation , I just use this method in a very particular way.

I'm sorry for my poor English :p.

Before we start , you must know follow:

- P an a_i are in $\mathbb{R}^n$, they are the same as which in $\mathbb{R}$
- 4 parameters can make certain a cubic line
- Linear Algebra

1. FERGUSON

Interpolation betwen 2 Point with cubic line.
let :

$$\begin{aligned} P(t) &\triangleq a_0 + a_1 t + a_2 t^2 + a_3 t^3 \\ &= \begin{pmatrix} 1 & t & t^2 & t^3 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} \end{aligned} \tag{1.1}$$

$$t \in [0,1] \tag{1.2}$$

So.

$$P'(t) = a_1 + 2a_2 t + 3a_3 t^2 \tag{1.3}$$

$$P(0) = a_0 \tag{1.4}$$

$$P(1) = a_0 + a_1 + a_2 + a_3 \tag{1.5}$$

$$P'(0) = a_1 \tag{1.6}$$

$$P'(1) = a_1 + 2a_2 + 3a_3 \tag{1.7}$$

Thus,

Robert A. Macy.

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix}^{-1} \begin{pmatrix} P(0) \\ P(1) \\ P'(0) \\ P'(1) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & -1 \\ 2 & -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} P(0) \\ P(1) \\ P'(0) \\ P'(1) \end{pmatrix} \tag{1.8}$$

$$\begin{aligned} P(t) &= \begin{pmatrix} 1 & t & t^2 & t^3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & -1 \\ 2 & -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} P(0) \\ P(1) \\ P'(0) \\ P'(1) \end{pmatrix} \\ &\triangleq \begin{pmatrix} F_0(t) & F_1(t) & G_0(t) & G_1(t) \end{pmatrix} \begin{pmatrix} P(0) \\ P(1) \\ P'(0) \\ P'(1) \end{pmatrix} \end{aligned} \tag{1.9}$$

F_0, F_1, G_0, G_1 is mix function:

$$\begin{aligned} F_0(t) &= 1 - 3t^2 + 2t^3 & F_1(t) &= 3t^2 - 2t^3 \\ G_0(t) &= t - 2t^2 + t^3 & G_1(t) &= -t^2 + t^3 \end{aligned} \tag{1.10}$$

2. Interpolation with 4 points

it is defficult to caulate P' , there is a way in the rough. Let:

$$P_i' = \alpha(P_{i+1} - P_{i-1}) \tag{2.1}$$

α a is a spline parameter such that $0 \leqslant a \leqslant 1$.

So, in order to interpolation between P_i and $P_i + 1$, we have to use P_{i-1} ,P_i ,P_{i+1} ,P_{i+2} , and :

$$\begin{aligned}
P_i(t) &= \begin{pmatrix} F_0(t) & F_1(t) & G_0(t) & G_1(t) \end{pmatrix} \begin{pmatrix} P_i \\ P_{i+1} \\ \alpha(P_{i+1} - P_{i-1}) \\ \alpha(P_{i+2} - P_i) \end{pmatrix} \\
&= \begin{pmatrix} F_0(t) & F_1(t) & G_0(t) & G_1(t) \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\alpha & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & \alpha \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{pmatrix} \\
&= \begin{pmatrix} -\alpha G_0(t) & F_0(t) - \alpha G_1(t) & F_1(t) - \alpha G_0(t) & \alpha G_1(t) \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{pmatrix} \\
&\triangleq \begin{pmatrix} C_0(t) & C_1(t) & C_2(t) & C_3(t) \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{pmatrix}
\end{aligned} \tag{2.2}$$

$C_0(t)$, $C_1(t)$,$C_2(t)$, $C_3(t)$ is the mix function.

$$C_0(t) = -\alpha t^3 + 2\alpha t^2 - \alpha t \tag{2.3}$$

$$C_1(t) = (2 - \alpha)t^3 + (\alpha - 3)t^2 + 1 \tag{2.4}$$

$$C_2(t) = (\alpha - 2)t^3 + (3 - 2\alpha)t^2 + \alpha t \tag{2.5}$$

$$C_3(t) = \alpha t^3 - \alpha t^2 \tag{2.6}$$

3. Border

To interpolation between $P_1, \ldots, P_n$, there is no P_0 and P_{n+1}, we have to set P_0 and P_{n+1} in reason.

let $P(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ is such cubic line. so $P'(t) = a_1 + 2a_2 t + 3a_3 t^2$.

$$P'(2) = P'_2 = \alpha(P_3 - P_1) \tag{3.1}$$

$$P(1) = P_1 \tag{3.2}$$

$$P(2) = P_2 \tag{3.3}$$

$$P(3) = P_3 \tag{3.4}$$

$$P(0) = P_0 \tag{3.5}$$

rewrite in matrix:

$$\begin{aligned}
\begin{pmatrix} 0 & 1 & 4 & 12 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} &= \begin{pmatrix} P'(2) \\ P(1) \\ P(2) \\ P(3) \end{pmatrix} \\
&= \begin{pmatrix} -\alpha & 0 & \alpha \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}
\end{aligned} \tag{3.6}$$

so:

$$P_0 = \begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}a_0\\a_1\\a_2\\a_3\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 1 & 4 & 12\\1 & 1 & 1 & 1\\1 & 2 & 4 & 8\\1 & 3 & 9 & 27\end{pmatrix}^{-1}\begin{pmatrix}-\alpha & 0 & \alpha\\1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}P_1\\P_2\\P_3\end{pmatrix} \tag{3.7}$$

$$= \begin{pmatrix}6(1-\alpha) & -3 & 6\alpha-2\end{pmatrix}\begin{pmatrix}P_1\\P_2\\P_3\end{pmatrix}$$

4. Bicubic Interpolation

In each , Bicubic is the same as cubic interpolation. So(easily):

$$P(s,t) = \begin{pmatrix}C_0(s) & C_1(s) & C_2(s) & C_3(s)\end{pmatrix}\begin{pmatrix}P_{i-1,j-1} & P_{i-1,j} & P_{i-1,j+1} & P_{i-1,j+2}\\P_{ij,-1} & P_{i,j} & P_{i,j+1} & P_{i,j+2}\\P_{i+1,j-1} & P_{i+1,j} & P_{i+1,j+1} & P_{i+1,j+2}\\P_{i+2,j-1} & P_{i+2,j} & P_{i+2,j+1} & P_{i+2,j+2}\end{pmatrix}\begin{pmatrix}C_0(t)\\C_1(t)\\C_2(t)\\C_3(t)\end{pmatrix} \tag{4.1}$$

Border also the same as cubic. For interpolation between $(P_{i,j})_{m\times n}$

$$P_{0,j} = 6(1-\alpha)P_{1,j} - 3P_{2,j} + (6\alpha-2)P_{3,j} \qquad j = 1,\cdots,n \tag{4.2}$$

$$P_{m+1,j} = 6(1-\alpha)P_{m,j} - 3P_{m-1,j} + (6\alpha-2)P_{m-2,j} \qquad j = 1,\cdots,n \tag{4.3}$$

$$P_{i,0} = 6(1-\alpha)P_{i,1} - 3P_{i,2} + (6\alpha-2)P_{i,3} \qquad i = 0,\cdots,m+1 \tag{4.4}$$

$$P_{i,n+1} = 6(1-\alpha)P_{i,n} - 3P_{i,n-1} + (6\alpha-2)P_{i,n-2} \qquad i = 0,\cdots,m+1 \tag{4.5}$$

5. The Program

In my Octave Program let $\alpha = \frac{1}{2}$ in default.
Why choose $\frac{1}{2}$ by default?
Consider this case: P_0, P_1, P_2, P_3 are on one line, and

$$P_0 - P_1 = P_1 - P_2 = P_2 - P_3 \tag{5.1}$$

and interprolation function is:

$$P(t) = \begin{pmatrix}C_0(t) & C_1(t) & C_2(t) & C_3(t)\end{pmatrix}\begin{pmatrix}P_0\\P_1\\P_2\\P_3\end{pmatrix} \tag{5.2}$$

follow equation is in natural.

$$P(t) = (1-t)P_1 + tP_2 \tag{5.3}$$

then we can calculate α by each of follow equation:

(5.4)

$$\begin{aligned}
\begin{pmatrix}1-t & t\end{pmatrix}\begin{pmatrix}P_1\\P_2\end{pmatrix} &= \begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix}\begin{pmatrix}1 & 0\\-1 & 1\\0 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}P_1\\P_2\end{pmatrix}\\
&= P(t)\\
&= \begin{pmatrix}C_0(t) & C_1(t) & C_2(t) & C_3(t)\end{pmatrix}\begin{pmatrix}P_0\\P_1\\P_2\\P_3\end{pmatrix}\\
&= \begin{pmatrix}C_0(t) & C_1(t) & C_2(t) & C_3(t)\end{pmatrix}\begin{pmatrix}2 & -1\\1 & 0\\0 & 1\\-1 & 2\end{pmatrix}\begin{pmatrix}P_1\\P_2\end{pmatrix}\\
&= \begin{pmatrix}2C_0(t)+C_1(t)-C_3(t) & -C_0(t)+C_2(t)+2C_3(t)\end{pmatrix}\begin{pmatrix}P_1\\P_2\end{pmatrix}\\
&= \begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix}\begin{pmatrix}1 & 0\\-2\alpha & 2\alpha\\6\alpha-3 & 3-6\alpha\\2-4\alpha & 4\alpha-2\end{pmatrix}\begin{pmatrix}P_1\\P_2\end{pmatrix}
\end{aligned}$$

treat P_1, P_2 and $1, t, t^2, t^3$ as bases , so :

$$\begin{pmatrix}1 & 0\\-1 & 1\\0 & 0\\0 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\-2\alpha & 2\alpha\\6\alpha-3 & 3-6\alpha\\2-4\alpha & 4\alpha-2\end{pmatrix} \tag{5.5}$$

thus, $\alpha = \frac{1}{2}$.

References

1. GengHong Liu, Differential geometry andCalculating geometry, 2000

Soochow University School of Mathematic Science
E-mail address: `Hoxide_dirac@yahoo.com.cn`